

ABOUT A MODULI SPACE OF ELLIPTIC CURVES AND THE GOLAY CODE G_{24}

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ABSTRACT. We investigate algebraic structures related to triangle decompositions of a moduli space of complex tori given by the Veech curve \mathbf{T}^* . We show that these structures produce the binary error correcting Golay code G_{24} .

1. INTRODUCTION

1.1. The Monster Group and VOA's. The relation between the monster group \mathbf{M} and the Klein modular function $J(\tau)$ still does not have an answer. The huge progress in this direction was made by Frenkel, Lepowski and Meurman construction [1] of the vertex operator algebra V^\sharp , i.e. the infinite dimensional graded representation of the sporadic group \mathbf{M} . This has made the connection between \mathbf{M} and the modular invariant J more natural. In other words, the fact that the graded dimension of V^\sharp is precisely equal to $J(\tau) - 744$ makes the monstrous moonshine module (the structure conjectured by Conway and Norton [2] and by Thompson [3]) less mysterious. It is important to notice that both objects: the monster group \mathbf{M} and the vertex operator algebra V^\sharp were developed from a sequence of structures that starts with the Golay error correcting code G_{24} and then continues with the Leech lattice Λ , a close relative of G_{24} and determined by it.

Lepowski and Meurman also gave an alternative construction that is based on three copies of the E_8 root lattice. This construction corresponds to the fact that the code G_{24} can be obtained from copies of some (unique up to isomorphism) Hamming code that further results in the construction of Λ as the non-orthogonal direct sum of three rescaled copies of the E_8 root lattice.

It is well known that to any even, unimodular lattice L with the automorphism group G we may associate a few new structures [4]. Thus, any such lattice L allows us to produce the following objects:

- Thompson series Θ_L where $\Theta_L^g = \sum \alpha_n(g) q^n$ for any element g in the automorphism group G .
- Thompson series Ω_G with $\Omega_g = \eta_g(q)$, where $\eta_g(q)$ is an appropriate, uniquely determined by an element g product of Dedekind eta functions. The ratio Θ_L/Ω_G forms a meromorphic Thompson series for G .
- Untwisted vertex operator algebra V_L
- A twisted vertex operator algebra V'_L .

The real task is to construct a vertex operator algebra associated to a lattice L that gives the moonshine module for the group that contains G or its particular extension. In our case, an even unimodular lattice is given as the Leech lattice Λ .

Date: December 14, 2010.

2010 Mathematics Subject Classification. Primary 30F99; Secondary 08A99.

The vertex operator algebra that forms monstrous moonshine is constructed as a \mathbb{Z}_2 -orbifold obtained from the torus \mathbb{R}^{24}/Λ . More precisely, the vertex operator algebra V^\sharp is equal to $V^\sharp = V_1^+ + V_1'^+$ where V_1 and V_1' denote untwisted and twisted VOA's associated to the Leech lattice Λ and the index “+” means that we consider only subspaces that are fixed by some involution in \mathbf{M} .

Although both, the investigations of Conway, Norton and Thompson (as well as many others) and the sophisticated constructions given by Frenkel, Lepowski and Meurman (and many others) are very beautiful, their connections to the Golay error correcting code G_{24} , which is the origin of all structures mentioned above, is still mysterious. In other words, the question: “What is the relation between the error correcting code G_{24} and elliptic curves ” has never been asked. We hope that in this paper we did answer this question.

1.2. Some Teichmueller Discs and Billiards. In our approach we will use the following relation between some subspace of the moduli space of a concrete compact Riemann surface of genus $g > 0$, say Σ , and the punctured surface Σ^* itself. This surprising fact results from both: the natural hyperbolic Poincare metric on Σ and some (singular) euclidean metric on it with appropriate symmetries. A flat structure with cone type singularities (when the genus is $g > 1$) on Σ is given by the horizontal and vertical trajectories of some holomorphic quadratic Jenkins-Strebel differential q on Σ , [5]. Moreover, the differential q determines a Teichmueller disc $\mathcal{D}_q(\Sigma)$ in the Teichmueller space \mathcal{T}_g . When q has symmetries that produce the stabilizer group \mathcal{G} of this disc in the full modular group \mathcal{M}_g of genus g , then the quotient of $\mathcal{D}_q(\Sigma)$ by the commutator subgroup $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ is exactly the punctured surface Σ^* itself. For example, let \mathbf{X}_n denote the hyperelliptic curve of genus $g = \frac{n-1}{2}$ that is given by the equation $y^2 = 1 - x^n$. The quadratic differential $q = (\frac{dx}{y})^2$ defines a flat metric on \mathbf{X}_n with cone type singularities when $g > 1$ as well as a Teichmueller disc $\mathcal{D}(\mathbf{X}_n)$ in \mathcal{T}_g with the origin in \mathbf{X}_n . Since q determines a decomposition of the surface \mathbf{X}_n into g vertical cylinders, the product of Dehn twists about the core loops of these cylinders forms a parabolic generator σ_n of the Schwarz triangle group $\mathcal{G}_n = \langle \sigma_n, \beta_n \rangle$, [6]. The second generator β_n also comes from the natural symmetries of the differential q . The stabilizer of this disc in the modular group \mathcal{M}_g is the Schwarz group \mathcal{G}_n . The compactification of the quotient of $\mathcal{D}(\mathbf{X}_n)$ by the commutator group \mathcal{G}'_n of \mathcal{G}_n is the Riemann surface \mathbf{X}_n^* again. In other words the punctured Riemann surface \mathbf{X}_n^* itself may be identified as a moduli space of some family of Riemann surfaces which also contains the surface \mathbf{X}_n itself. Thus we have :

$$\mathbf{X}_n \in \mathcal{D}(\mathbf{X}_n) \hookrightarrow \mathcal{T}_g$$

and

$$\mathcal{D}(\mathbf{X}_n) \rightarrow \mathcal{D}(\mathbf{X}_n)/\mathcal{G}'_n \cong \mathbf{X}_n^*$$

Pairs : [a compact Riemann surface, a holomorphic quadratic differential] that are of special importance to us are those coming from dynamical systems of billiards in rational polygons. Namely, the elastic reflections of a point-sphere particle moving with a constant velocity on the boundary of a polygon (i.e. a billiard) result in some identities. These identities glue an appropriate number of polygons together to form a compact Riemann surface that is equipped with a concrete flat metric structure whose cone type singularities occur at some vertices of glued polygons. The gluing process corresponds to the straightening out billiard trajectories. The genus of such

surface is determined merely by the shape of the original polygon [7]. When all angles of the polygon have the form $\frac{\pi}{k}$ with $k \in \mathbb{Z}^+$, then the billiard flow is without singular points and hence the corresponding Riemann surface must be a torus. Such case is realized only for billiards in rectangles and in triangles with angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ or $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ or $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$. Billiards in triangles $(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n})$ produce hyperelliptic Riemann surfaces \mathbf{X}_n together with their flat metric structure mentioned above and the group \mathcal{G}_n . The Schwarz groups \mathcal{G}_n are conjugated in $SL_2\mathbb{R}$ to the Hecke triangle groups \mathcal{H}_n generated by S and by h_n where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad h_n = \begin{pmatrix} 1 & 2\cos\frac{\pi}{n} \\ 0 & 1 \end{pmatrix}$$

The Hecke group \mathcal{H}_3 is exactly the modular group $\Gamma = SL_2\mathbb{Z}$ and the associated billiard is the $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ -triangle billiard which produces the torus $\mathbf{T} \cong \mathbf{X}_3$ obtained by gluing 12 such triangles together. The torus \mathbf{T} is the one point compactification of the quotient of the Teichmueller disc $\mathcal{D}(\mathbf{T})$ by the commutator subgroup $\Gamma' = [\Gamma, \Gamma]$. In this special case the disc $\mathcal{D}(\mathbf{T})$ is the whole Teichmueller space \mathcal{T}_1 and hence its quotient by the full modular group $\mathcal{M}_1 = \Gamma$ produces the whole modular space \mathbb{M}_1 for complex tori. Since the quotient of $\mathcal{D}(\mathbf{T})$ by Γ' produces \mathbf{T}^* , each point of it represents some complex torus and points in the same orbit of the automorphisms group of \mathbf{T}^* correspond to the isomorphic tori. In a generic case, any complex torus (i.e. any point of \mathbb{M}_1) is represented by 6 points of the Veech curve \mathbf{T}^* .

1.3. Some Additional Symmetries. The deep relation between the number theory and automorphic forms for different congruence subgroups of the modular group $\Gamma = SL_2\mathbb{Z}$ originates from the fact that the upper half-plane H (given by the set $\{\tau \in \mathbb{C}; \Im\tau > 0\}$) is the moduli space of the lattices $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$ in \mathbb{C} , of complex tori \mathbf{T}_τ (the quotient \mathbb{C}/L_τ) and of quadratic forms $Q_\tau(x, y) = \frac{1}{\Im\tau}|x + y\tau|^2$. All these objects are interrelated to each other and any $\gamma \in \Gamma$ acting on H produces the torus $\mathbf{T}_{\gamma\tau}$ that is isomorphic to \mathbf{T}_τ and the quadratic form $Q_{\gamma\tau}(x, y)$ that is equivalent to $Q_\tau(x, y)$.

For example, the quadratic form $Q_\tau(m, n)$ with $m, n \in \mathbb{Z}$ gives the lengths of closed geodesics on the torus \mathbf{T}_τ which leads to the geometric part of the problem (involving the counting of closed geodesics) and to the nonholomorphic Eisenstein series $E(\tau, s)$. Its closed relative $E^*(\tau, s)$ given by the product $\Gamma(s)\pi^{-s}E(\tau, s)$ is related to the theta series $\Theta_\tau(t)$ (associated to the quadratic form Q_τ) by the Mellin transform, etc.

The similar situation occurs when Σ is a compact Riemann surface of genus $g > 1$ equipped with a (singular) flat structure determined by some Jenkins-Strebel differential q with appropriate symmetries \mathcal{G} (as mentioned in the previous subsection). In this case, if α is a cusp of $\Sigma^* \cong \mathcal{D}_q(\Sigma)/\mathcal{G}'$ then the Eisenstein-Maas series $E_\alpha(z, s)$ (a pure hyperbolic entity) can be expressed in terms of the ‘single’ length spectrum of the closed leaves in the cylinders of a flat structure corresponding to (q, α) (a pure flat entity). Let us notice that we are dealing not only with the hyperbolic-euclidean duality. The group \mathcal{G}' (called the Veech group of (Σ, q)) not only has a finite covolume but it also has important dynamical properties associated to billiards in rational polygons. In this case we say that we have “Veech dichotomy” [8].

Before we will give an outline of this paper we would like to mention two different situations that are important for the comparison to our approach. First, we have the Millington approach ([9], [10]) which associates to a conjugacy class of subgroups of $PSL_2\mathbb{Z}$ a transitive permutation representation of $PSL_2\mathbb{Z}$. More precisely, if G is a subgroup of $\tilde{\Gamma} = PSL_2\mathbb{Z}$ of finite index n then the decomposition of a fundamental domain of G into n copies of a fundamental domains of $\tilde{\Gamma}$ leads to the permutation group $\Sigma(G) = \langle x, y \rangle < S_n$. This permutation group acts transitively on the set of n letters (set of cosets) and its generators $x; x^2 = 1$ and $y; y^3 = 1$ are the permutation images of the generators $S; \tau \rightarrow \frac{-1}{\tau}$ and $ST (T : \tau \rightarrow \tau + 1)$ of the modular group $\tilde{\Gamma}$ respectively. If a subgroup G of index n has signature $(g; e_2, e_3, t)$ then the permutation x fixes e_2 letters, permutation y fixes e_3 letters and their product xy consists of t disjoint cycles of lengths n_i with $i = 1, \dots, t$ corresponding exactly to the cusp split $n = \sum n_i$. The different choices of fundamental domains of the groups involved may produce transitive permutation groups whose generators \tilde{x}, \tilde{y} are necessarily simultaneously conjugate to the generators x, y . For example, for $G = \Gamma(2)$ and for its quadrilateral domain $\mathfrak{F}(\Gamma(2)) = (-1, 0, 1, \infty)$ we have $\mathfrak{F}(\Gamma(2)) = \bigcup_{\sigma \in \mathfrak{S}_1} \sigma F(\Gamma)$ where $F(\Gamma) = (i-1, \rho, i, \infty)$ and $\mathfrak{S}_1 = \{I, g, g^2, T, Tg, Tg^2\}$ with $g = ST$. When we choose the triangle $F_\Gamma = (0, \rho+1, \infty)$ as a fundamental domain for $\tilde{\Gamma}$ then we have $\mathfrak{F}(\Gamma(2)) = \bigcup_{\sigma \in \mathfrak{S}_2} \sigma F_\Gamma$ where $\mathfrak{S}_2 = \{I, a, a^2, S, Sa, Sa^2\}$ and $a = TS$. It is easy to check that in both cases the generators of the transitive permutation group $\Sigma(\Gamma(2)) < S_6$ are simultaneously conjugate to $x = (03)(14)(25)$ and to $y = (042)(351)$. Their product $xy = (01)(23)(45)$ corresponds to three cusps of width 2 each. On the other hand, taking the same quadrilateral domain $(-1, 0, 1, \infty)$ as a fundamental domain for the index 6 subgroup Γ' of the modular group $\tilde{\Gamma}$ will produce the permutation group $\Sigma(\Gamma') = \langle x', y' \rangle$ (determined uniquely up to a simultaneous conjugation in S_6) with $x' = (03)(14)(25)$, $y' = (042)(153)$ and with $x'y' = (012345)$ corresponding to the single cusp of width 6.

Another important situation occurs when a Riemann surface \mathbf{X} is defined over an algebraic number field, or equivalently, when there exists a Belyi function $\beta : \mathbf{X} \rightarrow P_1\mathbb{C}$ unbranched outside $\{0, 1, \infty\}$. The triangulation T_1 of $P_1\mathbb{C} \cong \hat{\mathbb{C}}$ (given by the three vertices $0, 1$ and ∞ ; the three edges along the line segment $\hat{\mathbb{R}}$ joining these vertices and two triangle faces given by the upper H^+ and the lower H^- half planes of $\mathbb{C} \subset \hat{\mathbb{C}}$ respectively) induces a triangulation $T = \beta^{-1}(T_1)$ of \mathbf{X} with $\beta^{-1}(H^+)$ and $\beta^{-1}(H^-)$ as open cells and with $\beta^{-1}\{0, 1, \infty\}$ as vertices. (It turns out that it is more convenient to work with the Grothendieck dessins $\beta^{-1}[0, 1]$ instead of with the whole $\beta^{-1}(T_1)$.) The monodromy group of a dessin is defined to be the monodromy group of the branching covering $\beta : \mathbf{X} \rightarrow \hat{\mathbb{C}}$. This is a two-generator transitive subgroup $\langle g_0, g_1 \rangle$ of the symmetric group S_N , where $N = \deg \beta$. To find the generators $g_i, i = 0, 1$ we look how all edges of the bipartite graph $\beta_{-1}[0, 1]$ incident with a unique vertex in $\{\beta^{-1}(i)\}$ are permuted by rotations around their common incident vertex. The cyclic ordering around vertices $\beta^{-1}(i), i = 0, 1$ form the disjoint cycles of the permutations g_i respectively [11]. Two dessins are isomorphic if and only if their monodromy generators are simultaneously conjugate. Since the absolute Galois group $Gal(\mathbb{Q}/\mathbb{Q})$ acts naturally on Belyi pairs (\mathbf{X}, β) we have its induced action on dessins. The purpose of those investigations is to find explicitly the structure of the absolute Galois group.

It is important to notice that to find the monodromy group of a dessin we may use instead of the bipartite graph $\beta^{-1}[0, 1]$ the introduced earlier triangulation T of \mathbf{X} . Now to find the generators g_1, g_2 of the transitive permutation group we use the orientation of \mathbf{X} and rotations of positive triangles around appropriate vertices [12].

We see that both, the Millington and the Belyi approaches consist of, roughly speaking, the following elements:

- Decompositions of appropriate fundamental domains into copies of a fundamental domain of the modular group Γ .
Or
- Decomposition of Riemann surfaces over algebraic number fields into positive ($\beta^{-1}(H^+)$) and negative ($\beta^{-1}(H^-)$) triangular cells.
And
- An action by some (external) operations on these elements.

In our approach we will start with the decompositions of fundamental domain $R = (\rho, \rho + 1, \infty)$ of Γ into two adjacent hyperbolic $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ triangles Δ_1 and Δ_2 and we will look for algebraic structures of the operations that produce all composite triangles of the standard fundamental domains for Γ' out of these two triangles. Since we obtain that the algebraic objects related to \mathfrak{F}'_4 are the same as those related to the quadrilateral domain for the congruence subgroup $\Gamma(2)$ (the differences between these two domains lie in the identifications on the borders) it is necessary to introduce also the hexagonal fundamental domain \mathfrak{F}'_6 for Γ' . Besides, the natural projection p of the Diag.2 maps \mathfrak{F}'_4 and \mathfrak{F}'_6 onto a fundamental parallelogram and the Brillouin zone of the lattice $L(\rho)$ and both of these regions are important. The reason for the consideration of both \mathfrak{F}'_4 and \mathfrak{F}'_6 is in fact much deeper and will be described in the next section.

In our approach we identify the punctured torus $\mathbf{T}^* \cong H/\Gamma'$ with the Riemann surface determined by the dynamical system of the billiard in \mathbf{P} and we consider the group Γ' as the Veech group of \mathbf{T} associated to its natural holomorphic quadratic differential. In other words we consider \mathbf{T}^* as a Veech modular space of complex tori (as described in the subsection (1.2)) instead of as (the more popular) base curve of the elliptic modular surface $B_{\Gamma'}$ (where $B_{\Gamma'}$ over $\mathbf{T}^* \cup \{\infty\}$ is semistable, of arithmetic genus zero, of geometric genus one with one singular fibre of type I_6^* , [13]). Since the fibre of $B_{\Gamma'}$ over the image in $\mathbf{T}^* \cong H/\Gamma'$ of a general point $\tau \in H$ is an elliptic curve corresponding to the lattice L_τ , the both realizations of \mathbf{T}^* (one given as the base curve of $B_{\Gamma'}$ and another given as a Veech curve) carry the same J -function.

Thus, similarly as in the Millington and Belyi cases we have decompositions of fundamental domains into subdomains. However our decompositions are into triangles and hence are more subtle than the Millington ones. The composite triangles can be viewed as coimages of the triangles H^+ and H^- of $\hat{\mathbb{C}}$ and hence are naturally divided into positive and negative as in Belyi approach. But in the contrary to the Millington and Grothendieck approaches our operations have totally internal nature. They merely produce all composite triangles of \mathfrak{F}'_4 and \mathfrak{F}'_6 out of the composite modular triangles Δ_1 and Δ_2 of R .

Now, viewing \mathbf{T}^* as a Veech curve associated to the dynamical system of the billiard in \mathbf{P} determines natural bijections σ , δ and κ on the set of appropriate euclidean triangles. Since these triangles (Pict.3) are the images of the hyperbolic

triangles (pict.1 and 2) respectively, we obtain the appropriate induced bijections between the hyperbolic triangle decompositions of \mathfrak{F}'_4 and \mathfrak{F}'_6 .

These algebraic operations defined on the set of our all 24 composite triangles $\Delta_i, i = 1, \dots, 24$ (either hyperbolic ones or their, determined by the Diag.2, euclidean images) allow us to associate to each triangle Δ_i a well defined subset S_i of triangles. In other words each composite triangle Δ_i is in natural algebraic relations with some unique subset S_i of triangles. When we represent these naturally arising subsets S_i by bit strings we obtain the generating matrix for the error correcting code G_{24} . Since the J -function determines the projection $J : \mathbf{T}^* \rightarrow Y(1) \cong H/\Gamma$ we may see this as some sort of a hidden structure associated to $J(\tau)$. (Some other relations between algebraic structures associated to Γ and Γ' are given in [14].)

As we have already mentioned in the section 1.1, the Golay code G_{24} is the starting point for the Leech lattice and then for the monster group \mathbf{M} . We have shown that for G_{24} to emerge we must consider a modular space given by the Veech curve \mathbf{T}^* instead of the full modular space \mathbb{M}_1 . This fact has its confirmation in the “hidden” relations between the Leech lattice Λ and elliptic curves with J -invariant zero discovered by Harada and Lang [15]. They have shown that all five curves associated to some special conjugacy classes of the Conway group $.O$ of all automorphisms of the Leech lattice are elliptic curves that represent the Riemann surface \mathbf{T}^* , that is, they all have J -invariant zero.

Let us notice that if we were able to associate the Veech curve \mathbf{T}^* to a real physical object then this object would be equipped with an error correcting code (i.e. it could correct itself) and hence it would behave as some sort of a biological object.

2. SOME ALGEBRAIC STRUCTURES

2.1. Fundamental Domains. The Teichmueller space $\mathcal{T}_1 \cong H$ of compact tori coincides with $\mathcal{T}_{1,1}$ of punctured tori. Thus, any $\tau \in H$ determines both: \mathbf{T}_τ and $\mathbf{T}_\tau^* \cong \mathbb{C} - L_\tau/L_\tau$. Any torus \mathbf{T}_τ has \mathbb{C} as its universal covering space (and hence it has a flat structure), whereas \mathbf{T}_τ^* is covered by the upper half-plane H with its natural Poincare hyperbolic metric. The group of deck transformations for the covering $\mathbb{C} \rightarrow \mathbf{T}_\tau$ is isomorphic to \mathbb{Z}^2 whereas in the latter case it is an appropriate Fricke group $\mathcal{F}_\tau \subset PSL_2(\mathbb{R})$, [16]. When $\tau = \rho = e^{\frac{2\pi i}{3}}$ the Fricke group \mathcal{F}_ρ is isomorphic to the commutator subgroup Γ' of the full modular group $\Gamma = SL_2\mathbb{Z}$ of genus one. Its standard fundamental region \mathfrak{F}'_4 coincide with the set of elements of the standard fundamental domain $\mathfrak{F}(\Gamma(2))$ for the congruence group $\Gamma(2)$. However, the identifications on $\partial\mathfrak{F}'_4$ are given by the generators $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ of $\Gamma' = \langle A, B \rangle$ and produce the punctured torus \mathbf{T}^* , whereas the identifications on $\partial\mathfrak{F}(\Gamma(2))$ are given by generators $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ of $\Gamma(2)$ and produce 3-punctured surface of genus zero.

Let \mathfrak{F} denote the quadrilateral $(-1, 0, 1, \infty)$ that is the underlying set for both \mathfrak{F}'_4 and $\mathfrak{F}(\Gamma(2))$. There are a few natural decompositions of the set \mathfrak{F} , each of which leads to some interesting structure. First, the imaginary axis divides \mathfrak{F} onto two ideal triangles (i.e. all vertices are at infinity). The group Γ acting on such ideal

triangle produces Farrey tassellation of H which, among others, allows us to present any geodesic of H by a continuous fraction decomposition using the symbolic dynamics [17]. The corresponding triangle group Δ_2^* (generated by the reflectios in the sides of such ideal triangle) has its associated group $\Delta(\infty, \infty, \infty)$ which is exactly equal to $\Gamma(2)$. We will not use this fact here. Some other decompositions which are given by $\mathfrak{F} = \mathfrak{S}_1 F(\Gamma)$ and $\mathfrak{F} = \mathfrak{S}_2 F_\Gamma$ were introduced earlier. (For $\Gamma(2)$ the quadrilateral fundamental domain is the most natural one since its reflects the fact that $\Gamma(2)$ is a Fuchsian triangle group $\Delta(\infty, \infty, \infty)$ and hence its fundamental domain is given by the union of a universal triangle and its adjacent one.) Since the modular group $PSL_2\mathbb{Z}$ is also a triangle Fuchsian group $\Delta(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ both its domains $F(\Gamma)$ and F_Γ are unions of two adjacent $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ -triangles and the standard fundamental domain R of $\tilde{\Gamma}$ is the union of $\Delta_1 = (\rho, i, \infty)$ and $\Delta_2 = (i, \rho + 1, \infty)$. However \mathfrak{F} cannot be written as $\bigcup_{k=1, \dots, 6} \gamma_k R$. Instead we have

$$\mathfrak{F} = \mathfrak{S}_1 \Delta_1 \cup \mathfrak{S}_2 \Delta_2$$

Let us enumerate the composite triangles of \mathfrak{F} as on Pict.1 ($\Delta_k \equiv k$).

Lemma 1. *The triangle decomposition of \mathfrak{F} introduced above has a natural algebraic structure of the cyclotomic coset decompositions of the quadratic residue \mathcal{Q} and nonresidue \mathcal{N} modulo 13 over \mathbb{F}_3 .*

Proof. We have already introduced two sets \mathfrak{S}_1 and \mathfrak{S}_2 of coset representatives for both, for Γ' and for $\Gamma(2)$ in $\tilde{\Gamma}$:

$$(2.1) \quad \mathfrak{S}_1 = \{T, TST, T(ST)^2\} \cup \{ST, (ST)^2, (ST)^3\}$$

and

$$(2.2) \quad \mathfrak{S}_2 = \{S, STS, S(TS)^2\} \cup \{TS, (TS)^2, (TS)^3\}$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. When we act on Δ_1 by the set \mathfrak{S}_1 of transformations and on Δ_2 by the transformations of \mathfrak{S}_2 we obtain all triangles of the decomposition. Since the set \mathcal{Q} of quadratic residue in \mathbb{F}_{13} is $\mathcal{Q} = \{1, 4, 3, 12, 9, 10\}$ and the set $\mathcal{N} = \{2, 8, 6, 11, 5, 7\}$ we see that $\mathfrak{S}_1(\Delta_1)$ corresponds to \mathcal{Q} and $\mathfrak{S}_2(\Delta_2)$ corresponds to \mathcal{N} precisely. Moreover the triangles Δ_k with $k \in \mathcal{Q}$ correspond to the positive triangular cells with respect to the both functions $J(\tau) : \mathfrak{F}'_4 \rightarrow \mathbb{C}$ and $\lambda(\tau) : \mathfrak{F}(\Gamma(2)) \rightarrow \mathbb{C}$ (i.e. their images under any of these mappings are given by the upper half-plane H^+ each). The images of the remaining triangles $\Delta_k, k \in \mathcal{N}$ are equal to the lower half-plane H^- each and hence their form the negative cells. Since we have

$$(2.3) \quad \begin{array}{ccccccccccc} & & \overset{3}{\curvearrowright} & & \overset{3}{\curvearrowright} & & \overset{3}{\curvearrowright} & & & & \\ \Delta_1 & \xrightarrow{T} & \Delta_4 & \xrightarrow{S} & \Delta_3 & \xrightarrow{T} & \Delta_{12} & \xrightarrow{S} & \Delta_9 & \xrightarrow{T} & \Delta_{10} & \xrightarrow{S} & \Delta_1 & \xrightarrow{T} & \Delta_4 \\ & & \underset{3}{\curvearrowleft} & & \underset{3}{\curvearrowleft} & & \underset{3}{\curvearrowleft} & & \underset{3}{\curvearrowleft} & & \underset{3}{\curvearrowleft} & & \end{array}$$

we may relate the decomposition (2.1) of \mathfrak{S}_1 to the multiplication of \mathcal{Q} by 3 and hence to the decomposition of \mathcal{Q} into disjoint union of cyclotomic cosets over \mathbb{F}_3

$$(2.4) \quad \mathcal{Q} = \{1, 3, 9\} \cup \{4, 12, 10\} = \mathcal{C}_1 \cup \mathcal{C}_4$$

Similarly, from the sequence

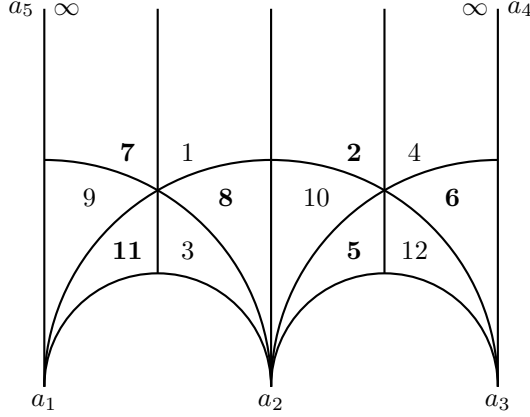
$$(2.5) \quad \Delta_2 \xrightarrow{S} \Delta_8 \xrightarrow{T} \Delta_6 \xrightarrow{S} \Delta_{11} \xrightarrow{T} \Delta_5 \xrightarrow{S} \Delta_7 \xrightarrow{T} \Delta_2 \xrightarrow{S} \Delta_8$$

we see that the decomposition (2.2) of \mathfrak{S}_2 corresponds to the multiplication of \mathcal{N} by 3 i.e. to the cyclotomic decomposition

$$(2.6) \quad \mathcal{N} = \{2, 6, 5\} \cup \{8, 11, 7\} = \mathcal{C}_2 \cup \mathcal{C}_7$$

□

From (2.3), (2.5) we observe that the operation of the multiplication of a composite triangle Δ_k by 3 corresponds to the action by the transformation $a = TS$ or by the transformation $g = ST$. The transformation a^m is equivalent modulo Γ' to the transformation g^m for $m = 1, 2, 3$ (more precisely we have $ST = B(TS) = (TS)A$ and $(ST)^2 = A^{-1}(TS)^2 = (TS)^2B^{-1}$) but these transformations are not $\Gamma(2)$ equivalent (a is equivalent to g^2 and a^2 is equivalent to g modulo $\Gamma(2)$). This means that our multiplication by 3 (i.e. sequences analogous to (2.3) and (2.5)) is well defined for $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ -triangle decomposition of any fundamental domain for Γ' but not for $\Gamma(2)$. The requirement of the quadrilateral domain $(-1, 0, 1, \infty)$ (for the operation of multiplication of composite triangles by 3) for $\Gamma(2)$ is a consequence of the nonabelian quotient $\tilde{\Gamma}/\Gamma(2) \cong S_3$.



Pict.1

Contrary to this, the subgroup Γ' is a character group of $\tilde{\Gamma}$ and hence there is no problem for the operation of multiplication by 3 to be well defined on any of its fundamental regions.

Now let us introduce the hexagonal fundamental domain \mathfrak{F}'_6 for Γ' together with its decomposition into hyperbolic $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ -triangles. The transformation T acting on this set of triangles has only two orbits which coincide with the decomposition of triangles into positive and negative cells determined by the function $J(\tau)$. So, T produces all positive triangles starting from one of them and the same is true for the

negative triangles. If we enumerate the positive triangles as before by elements of \mathcal{Q} and the negative triangles by elements of \mathcal{N} then the transformation T will correspond exactly to the multiplication by 4 in \mathbb{F}_{13}^* . In other words, take any positive triangle as Δ'_1 its adjacent to the right (negative) triangle as Δ'_2 and the triangle numeration is given by the identification of the transformation T with the multiplication of triangles indexes by 4 modulo 13 respectively. One such enumeration of hyperbolic $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ -triangles Δ'_k is shown on Pict.2.

Lemma 2. *The triangle decomposition of the hexagonal fundamental region \mathfrak{F}'_6 described above carries a natural structure of the multiplication by 4 of the quadratic residue \mathcal{Q} and nonresidue \mathcal{N} modulo 13.*

Proof. Again, the proof will be by construction. We observe that the action by T on the triangles Δ'_i is the following:

$$(2.7) \quad \Delta'_1 \xrightarrow{T} \Delta'_4 \xrightarrow{T} \Delta'_3 \xrightarrow{T} \Delta'_{12} \xrightarrow{T} \Delta'_9 \xrightarrow{T} \Delta'_{10} \xrightarrow{T} \Delta'_1$$

and

$$(2.8) \quad \Delta'_2 \xrightarrow{T} \Delta'_8 \xrightarrow{T} \Delta'_6 \xrightarrow{T} \Delta'_{11} \xrightarrow{T} \Delta'_5 \xrightarrow{T} \Delta'_7 \xrightarrow{T} \Delta'_2$$

Thus, after identifications of Δ'_i with i , the operation T coincides with the multiplication by 4 of the quadratic residue \mathcal{Q} i.e. to $1 \xrightarrow{4} 4 \xrightarrow{4} 3 \xrightarrow{4} 12 \xrightarrow{4} 9 \xrightarrow{4} 10 \xrightarrow{4} 1$ and to the multiplication by 4 of the quadratic nonresidue \mathcal{N} i.e. to $2 \xrightarrow{4} 8 \xrightarrow{4} 6 \xrightarrow{4} 11 \xrightarrow{4} 5 \xrightarrow{4} 7 \xrightarrow{4} 2$ respectively. \square

We notice immediately that although the sets of triangles $\{\Delta_k, k \in \mathcal{Q}\}$ and $\{\Delta'_k, k \in \mathcal{Q}\}$ are Γ' equivalent we do not have their pointwise equivalence. Similarly $\{\Delta_k, k \in \mathcal{N}\}$ and $\{\Delta'_k, k \in \mathcal{N}\}$ are Γ' equivalent (as the sets) but an element Δ_k is not necessarily Γ' equivalent to Δ'_k for $k \in \mathcal{N}$. To understand this better suppose that the triangle (ρ, i, ∞) (which is contained in both domains \mathfrak{F}'_4 and \mathfrak{F}'_6) has the same index i.e. that $\Delta_1 = \Delta'_1$. Since T^2 is Γ' -equivalent to the transformation g^2 we would have on the one side $T^2 \circ \Delta'_1 = 4^2 \Delta'_1 = \Delta'_3$ whereas $T^2 \circ \Delta_1$ is Γ' equivalent to $g^2 \circ \Delta_1 = \Delta_9$. So, since Δ'_k is not necessary Γ' equivalent to Δ_k we have chosen (for \mathfrak{F}'_6) the first positive triangle from the right as Δ'_1 and its adjacent (negative) as Δ'_2 . Using the transformation T (identified with with the operation of the multiplication by 4 modulo 13) we create all the remaining composite triangles of \mathfrak{F}'_6 . Let \hat{T} denote the set of transformations $\{T^k, k = 0, \dots, 5\}$. Thus $\mathfrak{F}'_6 = \hat{T}\Delta'_1 \cup \hat{T}\Delta'_2$ with $\hat{T}\Delta'_1 = \{\Delta'_k | k \in \mathcal{Q}\}$ and $\hat{T}\Delta'_2 = \{\Delta'_k | k \in \mathcal{N}\}$.

The lemmas 1 and 2 tell us that some algebraic informations contained in the sets of coset representatives \hat{T} and (for example) \mathfrak{S}_2 of Γ' in $PSL_2\mathbb{Z}$ are quite different. Thus, writing

$$\tilde{\Gamma}/\Gamma' \cong \langle S \rangle \times \langle g \rangle \cong \langle T \rangle \text{mod} T^6$$

we see that when we work with the quadrilateral domain \mathfrak{F}'_4 then, in fact we are dealing with the quotient $\tilde{\Gamma}/\Gamma'$ that may be identified with the direct product of the finite subgroups of $\tilde{\Gamma}$ itself. However, when we consider the hexagonal fundamental domain \mathfrak{F}'_6 then more natural is to view the quotient as given by $\langle T \rangle \text{mod} T^6$, with the infinite order element $T = Sg \in \tilde{\Gamma}$ being represented by the pair (S, g) . On the other hand, the elements $a = TS$ and $g = ST$ (which are of finite orders and which are Γ' equivalent) are responsible for the multiplication of (for example Δ_1) by 3

whereas the transformation T (which has infinite order in $\tilde{\Gamma}$) is responsible for the multiplication of (for example Δ_1) by 4. The former case provides a strong relation between Γ' and $\Gamma(2)$ whereas the latter one indicates strong relations between Γ' and the groups Γ_c and $\Gamma_{ns}^+(3)$, [14]. The genus zero subgroups Γ_c and $\Gamma_{ns}^+(3)$ of $\tilde{\Gamma}$ (which are associated to the non-split Cartan subgroups of $GL_2(2)$ and $GL_2(3)$ appropriately [18], [19]) have unique cusps of weight 2 and 3 respectively. Now, the branching schemes of the projections $\mathbf{X}' \rightarrow \mathbf{X}_c \rightarrow \mathbf{X}(1)$ and $\mathbf{X}' \rightarrow \mathbf{X}_{ns}^+(3) \rightarrow \mathbf{X}(1)$ over ∞ correspond exactly to the isomorphism $\langle T \rangle \text{mod} T^6 \cong \langle T^3 \rangle \times \langle T^2 \rangle \text{mod} T^6$. In other words we have

$$(2.9) \quad \mathfrak{F}'_6 = (I \cup T^3)F(\Gamma_{ns}^+(3)) = (I \cup T^2 \cup T^4)F(\Gamma_c)$$

where $F(G)$ denotes a fundamental domain of a Fuchsian group G and \mathbf{X}' , \mathbf{X}_c , $\mathbf{X}_{ns}^+(3)$ and $\mathbf{X}(1)$ are quotients of $H^* = H \cup \{\infty\} \cup \mathbb{Q}$ by the groups Γ' , Γ_c , $\Gamma_{ns}^+(3)$ and $\tilde{\Gamma}$ respectively.

Moreover, if J_2 denotes an absolute invariant of Γ_c and J_3 is an absolute invariant for $\Gamma_{ns}^+(3)$ then $\text{Span}_{\mathbb{C}}\{J_2, J_3\}$ forms the underlying vector space of very important (although a nonunitary) representation χ of $\tilde{\Gamma}$. This representation $\chi : PSL_2\mathbb{Z} \rightarrow GL_2\mathbb{C}$ is given by: $\chi(S) = I$ and by $\chi(T) = \begin{pmatrix} -1 & 0 \\ 0 & \rho \end{pmatrix}$. Since $\chi(S) = I$ and since J_2 and J_3 live also on \mathbf{T}^* and coincide with the Weierstrass functions \wp and \wp' there, the vector space $\text{Span}_{\mathbb{C}}\{J_2, J_3\} \cong \text{Span}_{\mathbb{C}}\{\wp, \wp'\}$, (see [14]) may be naturally viewed as the underlying space for the representation χ of the quotient $\tilde{\Gamma}/\Gamma'$ only when it is realized by $\langle T \rangle \text{mod} T^6$. So, it is the cusp of Γ' that is important here and it is the hexagonal domain \mathfrak{F}'_6 which immediately produces the relations (2.9).

Summarizing, our approach requires the introduction of both: the quadrilateral \mathfrak{F}'_4 and the hexagonal \mathfrak{F}'_6 fundamental domains for Γ' . In both cases the subsets of the composite triangles enumerated by the elements of \mathcal{Q} or of \mathcal{N} are Γ' equivalent respectively but the particular enumeration of the triangles of \mathfrak{F}'_4 and of \mathfrak{F}'_6 have to be independent. On the composite triangles of \mathfrak{F}'_4 we have naturally defined the operation of multiplication by 3 modulo 13 whereas the natural operations on the composite triangles of \mathfrak{F}'_6 is multiplication by 4 modulo 13. Both these operations have “internal” nature. To distinguish between these two “memberships” we have introduced the triangles Δ_i and Δ'_i respectively. From now on, the triangles Δ'_i will be denoted by Δ_{12+i} . An introduction of any further algebraic structure will keep this notations fixed.

In the next subsection we will use the fact that the quadrilateral domain \mathfrak{F}'_4 is mapped into a fundamental parallelogram of the lattice $L(\rho)$ in \mathbb{C} with a vertex at the origin and with the border which forms a locus of real values of $\wp'(z, L(\rho))$. The hexagonal domain \mathfrak{F}'_6 of Γ' is mapped into the Brillouin zone for the lattice $L(\rho)$ centered at $0 \in \mathbb{C}$. Its border forms a locus of pure imaginary values of $\wp'(z, L(\rho))$. It is well known that both of these regions are of a great importance.

Now, when we consider $\mathbf{X}' = \mathbf{T}^* \cup \infty \cong H^*/\Gamma'$ then the image $\hat{\Delta}$ of any $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ -hyperbolic triangle of the tassellation of H given by $\Gamma'(\rho, i, \infty)$ must have the double “memberships” which indicates whether \mathbf{T}^* is obtained from \mathfrak{F}'_4 or from \mathfrak{F}'_6 by appropriate identifications on their borders.

2.2. Bijections Introduced by the Billiard Dynamics. In a general case, when \mathcal{F}_τ is the Fricke group corresponding to a punctured torus \mathbf{T}^*_τ we have the following

commutative diagram:

$$\begin{array}{ccccc}
 & & \xrightarrow{r} & & \\
 & \swarrow & & \searrow & \\
 H & \xrightarrow{p_\tau} & H/N_\tau & \longleftrightarrow & \mathbb{C} - L_\tau \\
 \pi_\tau \downarrow & & p'_\tau \downarrow & & \downarrow \\
 \mathbf{T}_\tau^* & \longleftrightarrow & H/\mathcal{F}_\tau & \longleftrightarrow & \mathbb{C} - L_\tau/L_\tau
 \end{array}$$

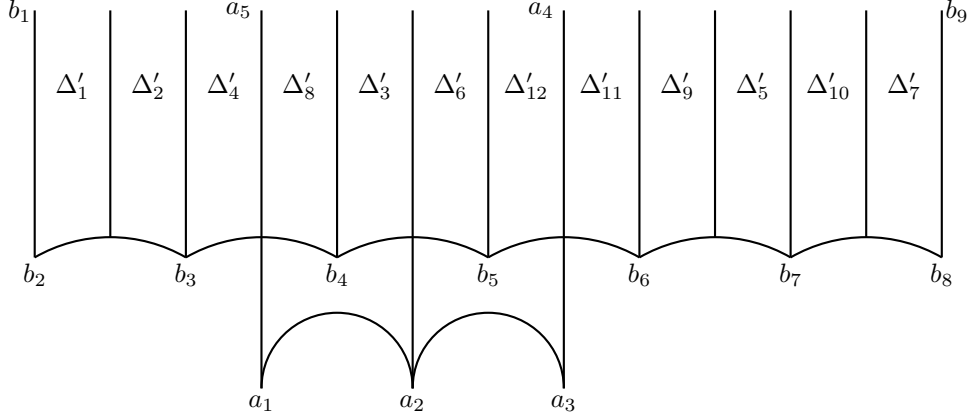
Diag.1

Here p_τ , p'_τ and π_τ are the natural projections, r is a conformal mapping (local inverse of p_τ) which satisfies $\{r, z\} = \frac{1}{2}\wp(z, \tau)$ with $\{\}$ denoting the Schwarzian derivative. $N_\tau = [\mathcal{F}_\tau, \mathcal{F}_\tau]$ and $\mathcal{F}_\tau/N_\tau \cong \mathbb{Z}^2$. When τ is equal to $\rho = e^{\frac{2\pi i}{3}}$ then $\mathbf{T}_\rho^* = \mathbf{T}^*$ and Diag.1 can be extended to the following one:

$$\begin{array}{ccccc}
 & & \xrightarrow{r} & & \\
 & \swarrow & & \searrow & \\
 H & \xrightarrow{p} & H/N & \longleftrightarrow & \mathbb{C} - L(\rho) \\
 \pi \searrow & & p' \downarrow & & \downarrow \\
 & & \mathbf{T}^* & \longleftrightarrow & \mathbb{C} - L(\rho)/L(\rho) \\
 J \downarrow & & \swarrow & & \downarrow \phi'^2 \\
 \mathbb{C} & \xleftarrow{T_1} & \mathbb{C} & &
 \end{array}$$

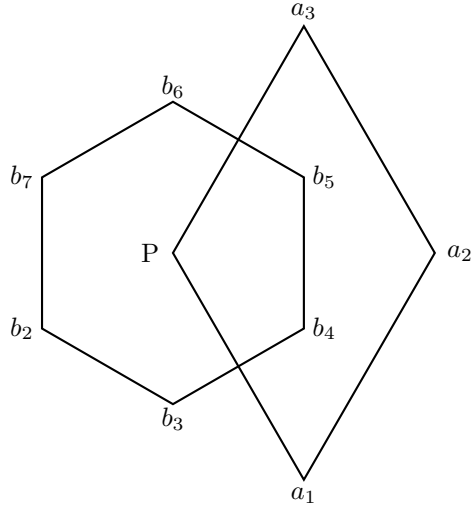
Diag.2

Here $N = [\Gamma', \Gamma']$ and $T_1 : z \rightarrow z + 1$. It is known that the mapping $p : H \rightarrow \mathbb{C} - L(\rho)$ maps the quadrilateral fundamental domain \mathfrak{F}'_4 in H onto the fundamental parallelogram F_4 in the z plane $\mathbb{C} - L(\rho)$ and the hexagonal fundamental domain \mathfrak{F}'_6 onto the regular hexagon F_6 in the z -plane [16]. This is illustrated on the Pict.2 and on Pict.3 respectively. Vertices $a_1 = -1, a_2 = 0, a_3 = 1$ in the extended upper half-plane $H^* = H \cup \mathbb{Q} \cup \{\infty\}$ are translates of ∞ by elements of Γ' and are identified by the Γ' generators A and B . The mapping p maps the Γ' translates of ∞ into the vertices of $L(\rho)$ and the points a'_i s of H^* onto the vertices of the fundamental parallelogram of the lattice $L_0 = L(\rho) \cong L_\rho$ in \mathbb{C} . Moreover, $i \in H$ and its Γ translates are mapped onto half points of $L(\rho)$. On $\partial\mathfrak{F}'_4$ the identifications are as follows: $a_5a_1 \xrightarrow{A} a_3a_2$ with $A(a_5) = a_3$ and $A(a_1) = a_2$; $a_4a_3 \xrightarrow{B} a_1a_2$ with $B(a_4) = a_1$ and $B(a_3) = a_2$ (here $a_4 = a_5 = \infty$). On $\partial\mathfrak{F}'_6$ the identifications are: $b_2b_3 \xrightarrow{A} b_6b_5$ with $A(b_2) = b_6$, $A(b_3) = b_5$; $b_6b_7 \xrightarrow{B} b_4b_3$ with $B(b_6) = b_4$ and $B(b_7) = b_3$; $b_4b_5 \xrightarrow{C} b_8b_7$ with $C(b_4) = b_8$ and $C(b_5) = b_7$ (here $b_1 = b_9 = \infty$ and $C = B^{-1}A^{-1}$). These all identifications induce the appropriate identifications on ∂F_4 and ∂F_6 respectively. The decompositions of \mathfrak{F}'_4 and \mathfrak{F}'_6 onto hyperbolic $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ triangles determine the decompositions (together with their enumeration) of F_4 and F_6 into euclidean $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangles respectively.



Pict.2

Now, the embeddings of F_4 and F_6 into \mathbb{C} and their decompositions into $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ euclidean triangles correspond to the tassellation of \mathbb{R}^2 by the copies of the triangle billiard in the triangle \mathbf{P} . The process of gluing together 12 copies of the triangle \mathbf{P} yields two (equivalent) tassellations of \mathbb{R}^2 given by the period parallelogram $F_4 = (a_1, a_2, a_3, a_4)$ or by the regular hexagon $F_6 = (b_2, b_3, \dots, b_7)$ respectively (see Pict.3).

Pict.3 $(P = a_4 = b_9)$

The process of straightening out billiard's trajectories corresponds to reflections of the triangle \mathbf{P} (or its copies) about its edges. In other words, the projection of any line $l \subset \mathbb{R}^2$ onto \mathbf{P} forms a billiard trajectory, whereas the projection of l onto F_4 or F_6 (using appropriate identifications on the borders) is a flat geodesics in \mathbf{X}' . When we start with F_4 then the dynamics of the billiard in \mathbf{P} corresponds to reflections of the composite triangles about the border of F_4 and hence to the reflections of the

euclidean $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangles in their longer perpendicular. If we start with F_6 then the “billiard identifications” at its border correspond to the reflections of a composite triangle in its shorter perpendicular. Hence, the billiard dynamics introduces the following bijections on the sets of the composite triangles of F_4 and F_6 respectively:

- The bijection σ , which corresponds to the reflections of $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangles in their longer perpendicular.
- The bijection δ that describes the reflections of the composite triangles in their shorter perpendicular.
- The bijection κ , the natural bijection between the sets of composite triangles of F_4 and F_6 .

More precisely, the action of σ is given by (we use the notation k for Δ_k):

$$\begin{array}{ll}
 1 \rightarrow 8 \rightarrow 1 & 13 \rightarrow 23 \rightarrow 13 \\
 2 \rightarrow 10 \rightarrow 2 & 14 \rightarrow 24 \rightarrow 14 \\
 3 \rightarrow 11 \rightarrow 3 & 16 \rightarrow 17 \rightarrow 16 \\
 4 \rightarrow 12 \rightarrow 4 & 20 \rightarrow 21 \rightarrow 20 \\
 5 \rightarrow 12 \rightarrow 5 & 15 \rightarrow 19 \rightarrow 15 \\
 7 \rightarrow 9 \rightarrow 7 & 18 \rightarrow 22 \rightarrow 18
 \end{array}$$

Similarly, for the mapping δ we have:

$$\begin{array}{ll}
 13 \rightarrow 14 \rightarrow 13 & 4 \rightarrow 11 \rightarrow 4 \\
 15 \rightarrow 18 \rightarrow 15 & 5 \rightarrow 9 \rightarrow 5 \\
 16 \rightarrow 20 \rightarrow 16 & 6 \rightarrow 3 \rightarrow 6 \\
 17 \rightarrow 21 \rightarrow 17 & 7 \rightarrow 12 \rightarrow 7 \\
 19 \rightarrow 22 \rightarrow 19 & 8 \rightarrow 10 \rightarrow 8 \\
 23 \rightarrow 24 \rightarrow 23 & 1 \rightarrow 2 \rightarrow 1
 \end{array}$$

Of course we have the analogous bijections (also denoted by σ and δ) on the set of the composite triangles of \mathfrak{F}'_4 and \mathfrak{F}'_6 . In other words, the above relations may be read from the reflections of the composite hyperbolic triangles in the borders $\partial\mathfrak{F}'_4$ and $\partial\mathfrak{F}'_6$ using the borders identifications given by A and B or by A , B and $C = B^{-1}A^{-1}$ respectively. However, reflections are natural for billiards and do not belong to the hyperbolic “picture” of \mathbf{T}^* (they belong to the Veech “picture” instead).

The last bijection κ [which also may be read from either the hyperbolic, Pict.1 and Pict.2, or from the billiard relations] is the following:

$$\begin{array}{ll}
 1 \leftrightarrow 15 & 7 \leftrightarrow 20 \\
 2 \leftrightarrow 18 & 8 \leftrightarrow 19 \\
 3 \leftrightarrow 13 & 9 \leftrightarrow 21 \\
 4 \leftrightarrow 24 & 10 \leftrightarrow 22 \\
 5 \leftrightarrow 17 & 11 \leftrightarrow 23 \\
 6 \leftrightarrow 14 & 12 \leftrightarrow 16
 \end{array}$$

Let Ω denote the set of 24 triangles Δ_i 's such that $\bigcup_{i=1}^{12} \Delta_i = F_4 \cong \mathfrak{F}'_4$ and $\bigcup_{i=13}^{24} \Delta_i = F_6 \cong \mathfrak{F}'_6$.

3. MODULI SPACE AND THE GOLAY CODE

3.1. Structures on the Set Ω . We have seen that the hyperbolic structures on \mathfrak{F}'_4 and on \mathfrak{F}'_6 induces the following:

- On the set of composite triangles of F_4 the natural operation of multiplication by 3 (equivalently, the cyclotomic coset structure of the quadratic residue and nonresidue in \mathbb{F}_{13} over \mathbb{F}_3).
- On the set of composite triangles of the regular hexagon F_6 the natural operation of multiplication by 4.
- The natural bijection κ between the sets of the composite triangles of these two regions.

On the other hand the flat structure on \mathbf{T}^* determined by the dynamical system of the billiard produces the mappings σ and δ defined in the previous subsection. When a $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ -triangle has the F_4 membership (i.e. for Δ_i with $i = 1, \dots, 12$) the natural operation is the multiplication by 3 and the natural mapping is σ (since it is introduced by the billiard reflections in the border ∂F_4). When a triangle has the F_6 membership (i.e. for Δ_i with $i = 13, \dots, 24$) the natural operation is the multiplication by 4 and the natural mapping is δ (corresponding to the billiard reflections in ∂F_6). Thus, on the set $\Omega = \{\Delta_i | i = 1, \dots, 24$ we have rather rich algebraic properties. They allow us to construct natural correspondences which to each $\Delta_i \equiv i$ associate a subset $S_i \subset \Omega$ in the following way: For $i = 1, \dots, 12$

$$(3.1) \quad \begin{array}{ccccc}
 [3i] & \xleftarrow{3} & i & \xrightarrow{\sigma} & \sigma(i) & \xrightarrow{3} & [3\sigma(i)] \\
 & & \updownarrow \kappa & & \updownarrow \kappa & & \\
 4 \circ \kappa(i) & \xleftarrow{4} & \kappa(i) & \xrightarrow{\sigma} & \sigma\kappa(i) & \xrightarrow{4} & 4 \circ \sigma\kappa(i)
 \end{array}$$

Here the square bracket denotes the congruence modulo 13, 3 and 4 denote the multiplications by 3 and by 4 modulo 13 respectively. Moreover, since we have to make appropriate adjustments caused by the identification of Δ'_i with Δ_{i+12} , we introduce

$$4 \circ \kappa(i) := [4(\kappa(i) - 12)] + 12$$

and

$$4 \circ \sigma\kappa(i) := [4(\sigma\kappa(i) - 12)] + 12$$

Notice that instead of using the language of quadratic residue and nonresidue of \mathbb{F}_{13} , cyclotomic cosets and so on we could consider appropriate sequences of transformations S and T acting on the hyperbolic triangle decompositions of \mathfrak{F}'_4 and \mathfrak{F}'_6 appropriately. However, these elementary number theory representations make our formulae much more transparent and elegant. For example, for $i = 1$ we have

$$(3.2) \quad \begin{array}{ccccc} 3 & \xleftarrow{3} & \mathbf{1} & \xrightarrow{\sigma} & 8 & \xrightarrow{3} & 11 \\ & & \updownarrow \kappa & & \updownarrow \kappa & & \\ 24 & \xleftarrow{4} & 15 & \xrightarrow{\sigma} & 19 & \xrightarrow{4} & 14 \end{array}$$

Hence, for the triangle Δ_1 the associated subset S_1 of the set Ω is given by $\{1, 8, 3, 11, 15, 19, 24, 14\}$. Sorting the elements of each set S_k in their increasing order we obtain the following:

$$\begin{aligned} S_1 &= \{1, 3, 8, 11, 14, 15, 19, 24\} = S_8 \\ S_2 &= \{2, 4, 6, 10, 13, 18, 22, 23\} = S_{10} \\ S_3 &= \{3, 7, 9, 11, 13, 16, 17, 23\} = S_{11} \\ S_4 &= \{4, 5, 6, 12, 14, 20, 21, 24\} = S_6 \\ S_5 &= \{2, 5, 10, 12, 15, 16, 17, 19\} = S_{12} \\ S_7 &= \{1, 7, 8, 9, 18, 20, 21, 22\} = S_9 \end{aligned}$$

Since the bijections σ , δ and κ satisfy:

$$(3.3) \quad \kappa\sigma = \sigma\kappa \quad \text{and} \quad \kappa\delta = \delta\kappa$$

we see immediately that $S_i = S_{\sigma(i)}$. Similarly, to each triangle Δ_j with $j = 13, \dots, 24$ we associate a subset S_j according to:

$$(3.4) \quad \begin{array}{ccccccc} 4 \circ j & \xleftarrow{4} & \mathbf{j} & \xrightarrow{\delta} & \delta(j) & \xrightarrow{4} & 4 \circ \delta(j) \\ & & \updownarrow \kappa & & \updownarrow \kappa & & \\ [3\kappa(j)] & \xleftarrow{3} & \kappa(j) & \xrightarrow{\delta} & \kappa\delta(j) & \xrightarrow{3} & [3\kappa\delta(j)] \end{array}$$

where as above $4 \circ j := [4(j - 12)] + 12$ and $4 \circ \delta(j) := [4(\delta(j) - 12)] + 12$ correct our notation. Thus, for example for $j = 13$ we have:

$$(3.5) \quad \begin{array}{ccccccc} 16 & \xleftarrow{4} & \mathbf{13} & \xrightarrow{\delta} & 14 & \xrightarrow{4} & 20 \\ & & \updownarrow \kappa & & \updownarrow \kappa & & \\ 9 & \xleftarrow{3} & 3 & \xrightarrow{\delta} & 6 & \xrightarrow{3} & 5 \end{array}$$

Again we have $S_j = S_{\delta(j)}$ for $j = 13, \dots, 24$ and sorting their elements by the increasing order the explicit forms of these subsets of Ω are:

$$\begin{aligned} S_{13} &= \{3, 5, 6, 9, 13, 14, 16, 20\} = S_{14} \\ S_{15} &= \{1, 2, 3, 6, 15, 18, 23, 24\} = S_{18} \\ S_{16} &= \{7, 8, 10, 12, 15, 16, 18, 20\} = S_{20} \\ S_{17} &= \{1, 2, 5, 9, 17, 19, 21, 22\} = S_{21} \\ S_{19} &= \{4, 8, 10, 11, 13, 14, 19, 22\} = S_{22} \\ S_{23} &= \{4, 7, 11, 12, 17, 21, 23, 24\} = S_{24} \end{aligned}$$

Now, each subset S_i , $1 = 1, \dots, 24$ may be represented by a bit string of the length 24. Passing to the bit string description we immediately notice that, for example S_7 is equal to the sum of the strings $S_1 + S_2 + S_3 + S_4 + S_5$ and that S_{23} is equal to the sum $S_{13} + S_{15} + S_{16} + S_{17} + S_{19}$. This is not surprising. From the correspondences $\Delta_i \rightarrow S_i$ it is obvious that the association $\Delta_7 \rightarrow S_7$ as well as $\Delta_{23} \rightarrow S_{23}$ are already described by the remaining correspondences. However, we still did not involve the explicit correspondences between the cyclotomic coset decompositions and the composite triangles.

So, to have the full description of algebraic structures associated to the Teichmüller disc $\mathcal{D}(\mathbf{T}) \cong H$ with symmetries determined by the appropriate holomorphic quadratic differential (equivalently by the dynamical system of the billiard in \mathbf{P}) we must also consider the following subsets of Ω :

$$(3.6) \quad R_6 := \mathcal{C}_1 \cup \mathcal{C}'_4 \cup \mathcal{C}'_2 \cup \mathcal{C}_7$$

and

$$(3.7) \quad R_{12} := \mathcal{C}'_1 \cup \mathcal{C}_4 \cup \mathcal{C}_2 \cup \mathcal{C}'_7$$

Here $\mathcal{C}_1 = \{1, 3, 9\}$ and $\mathcal{C}_4 = \{4, 12, 10\}$ are cyclotomic cosets of \mathcal{Q} over \mathbb{F}_3 whereas $\mathcal{C}_2 = \{2, 6, 5\}$ and $\mathcal{C}_7 = \{8, 11, 7\}$ are the cyclotomic cosets of \mathcal{N} . The κ -image of a coset \mathcal{C}_l is denoted by \mathcal{C}'_l .

Geometrically, the subsets R_6 and R_{12} are associated to the decomposition of \mathfrak{F}'_4 onto two ideal triangles. More precisely, R_6 represents all composite triangles of the ideal hyperbolic triangle with the vertices at $(\infty, -1, 0)$ and the κ -images of the composite triangles of the other half of \mathfrak{F}'_4 (i.e. of the ideal triangle $(0, 1, \infty) \subset \mathfrak{F}'_4$). The subset R_{12} represents vice versa, all composite triangles of $(0, 1, \infty)$ and the κ -images of $(\infty, -1, 0)$. We may say, that similarly as the necessity of the involving of the hexagonal region for Γ' is connected to the relation between the groups Γ' and the mentioned earlier $\Gamma_{ns}^+(3)$ and Γ_c the subsets given by the formulas 3.6 and 3.7 are connected to the relations between Γ' and $\Gamma(2)$.

Our original enumeration of the composite hyperbolic $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ triangles could be quite different. Although the decompositions into positive (corresponding to the set of elements of \mathcal{Q}) triangles and negative (corresponding to the set of elements of \mathcal{N}) triangles are fixed (by the values of the modular invariant $J(\tau)$) we could take any positive triangle in \mathfrak{F}'_4 as Δ_1 and its adjacent as Δ_2 and create the remaining composite triangles of the quadrilateral using (2.3) and (2.5). Similarly any positive triangle of \mathfrak{F}'_6 and its adjacent (which is of course negative) can be taken as $\Delta'_1 = \Delta_{13}$ and $\Delta'_2 = \Delta_{14}$ respectively and the remaining composite triangles can

be obtain from them using (2.7) and (2.8). This means that our relations $\Delta_i \rightarrow S_i$ could look quite differently. However, the algebraic structures which are determined by the sequences (2.3), (2.5), (2.7) and (2.8) as well as by the properties of the mappings σ , δ and κ would be exactly the same, merely described by a different notation.

Summarizing, all natural algebraic properties that may be associated to \mathbf{T}^* seen as a Veech moduli space of compact complex tori and related to its hyperbolic and euclidean structures, are totally described by the following set of subsets of Ω :

$$(3.8) \quad \{S_1, S_2, S_3, S_4, S_5, S_{13}, S_{15}, S_{17}, S_{19}\} \cup \{R_6, R_{12}\}$$

3.2. Moduli Space and the Golay Code. In the previous subsections we have obtained same algebraic structures that uniquely describe the nature of the triangle decompositions of the quadrilateral and the hexagonal fundamental regions of Γ' together with their relations to the billiard's dynamics in \mathbf{P} . However, instead of working with these two fundamental domains we could choose only one of them and consider each of its composite triangle twice (i.e. give each of them two, independent indices) depending whether it undergoes the operation of multiplication by 3 and the operation σ or it undergoes the operation of multiplication by 4 and by δ . Thus each composite triangle of any fundamental domain for Γ' would have two (not necessarily equal) labels connected by the obvious bijection κ . Of course, the algebraic structure which describes all algebraic-geometric relations between these 24 (single) labelled triangles is again given by the set of the form of (3.8).

Proposition 1. *The algebraic structure mentioned above and describing all algebraic informations given by the triangle decompositions of the fundamental parallelogram and the regular hexagon (viewed as the p -images of the domains \mathfrak{F}'_4 and \mathfrak{F}'_6 of Γ') is exactly the one given by the error correcting Golay code G_{24} .*

Proof. First, let us rename our subsets S_i 's as follows: $R_i := S_i$ for $i = 1, \dots, 5$. Let $R_7 := S_{13}$, $R_8 := S_{15}$, $R_9 := S_{16}$, $R_{10} := S_{17}$ and let $R_{11} := S_{19}$. Now it is the set $\{R_i | i = 1, \dots, 12\}$ that contains all informations about the algebraic properties carried out by the triangle decompositions of the moduli space \mathbf{T}^* . Since each subset R_i corresponds to a bit string we may construct a matrix \mathbf{G} whose i -th row is given by the bit string of length 24 determined by R_i .

$$(3.9) \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

These bit strings span 12-dimensional subspace of the space \mathbb{F}_2^{24} . It was checked (on a computer) that the weight distribution is given exactly by

$$(3.10) \quad 1 + 759q^8 + 2576q^{12} + 759q^{16} + q^{24}$$

We recall that the weight of a bit string is given by the number of 1's. Now, using well known theorem (see [20]) we obtain that our set $\{R_1, \dots, R_{12}\}$ determines unique, up to isomorphism, error correcting binary Golay code G_{24} . \square

We notice that our generating matrix \mathbf{G} for the binary code G_{24} has quite different form than the standard one. The reason for this is that usually to construct some code we are trying to find a nice matrix (i.e. a subspace of \mathbb{F}_k^n) whereas here our matrix comes from the algebraic structures naturally arising from the (doubly indexed) triangle decompositions of the moduli space (given by the Veech space \mathbf{T}^*) of compact complex tori. Since the modular invariant $J(\tau)$ lives on this moduli space and the projection $J: \mathbf{T}^* \rightarrow Y(1)$ determines the decompositions of \mathbf{T}^* into all positive (\mathcal{Q}) and negative (\mathcal{N}) triangle cells we have obtained a sort of a hidden structure which is given by G_{24} and which is associated to $J(\tau)$. By an appropriate change of the enumeration of our triangles we may get the standard form of the generating matrix for the Golay code G_{24} .

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